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LETTER TO THE EDITOR

A locally supersymmetric and reparametrisation invariant action for a spinning membrane

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Abstract. We present an action for an extended two-dimensional system which possesses, in addition to reparametrisation invariance, local supersymmetry. It is therefore a candidate to generate the theory of an intrinsically spinning membrane.

There has been a great deal of activity recently in the field of local supersymmetry (Deser and Zumino 1976a, Ferrara *et al* 1976). The subject has not only excited interest in those concerned with the renormalisability of quantum gravity but it has also been of considerable use in constructing 'classical' theories of spinning particles (Brink *et al* 1976a, 1977) and strings (Brink *et al* 1976b, Deser and Zumino 1976b). In these cases the Grassmann algebra structure of the theories becomes upon quantisation the corresponding Dirac–Clifford algebra.

Some years ago Dirac (1962) proposed a theory of an extended electron based on the idea of a relativistic membrane, i.e. a closed two-dimensional surface moving through space–time. More recently, the topic has been discussed in some detail (Collins and Tucker 1976a) as an extension of the string picture of elementary particles. Dirac did not endow his extended electron with intrinsic spin and it is only recently that spin has been incorporated into 'classical' particle and string actions (Brink *et al* 1976a, b, 1977, Deser and Zumino 1976b, Collins and Tucker 1976b, 1977, Cassalbuoni 1976a, b, Berezin and Marinov 1975, 1976). The essential idea required to describe intrinsic spin in these cases was the recognition that the constraint structure could be generated from a locally supersymmetric action. In Collins and Tucker (1976b, 1977) an economical description resulted with the aid of supernumerary variables. However, by approaching the problem as a field theory (in one or two dimensions) it was possible to utilise the techniques developed in four dimensions for supergravity. In this way the gauge fields acquired an elegant geometrical interpretation. In this letter we show that a similar procedure may be applied to the theory of the relativistic membrane.

As with the spinless string we can describe the ordinary membrane (without intrinsic spin) in terms of a three-dimensional field theory. Indeed, the volume element action (Collins and Tucker 1976a) may be recast in the form

$$S=-\tfrac{1}{2}\int d^3x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-1) \tag{1}$$

where $g_{\mu\nu}(x)$ is the three-dimensional metric tensor and the set of scalar fields $\phi^\alpha(x)$ ($\alpha=0,1,2,3$) locate the world tube swept out by the membrane in space–time.

Whenever bilinear combinations of the ϕ appear we imply a (3+1) Lorentz scalar. Varying (1) with respect to ϕ^α yields the Euler–Lagrange equations

$$\Box\phi^\alpha \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial_\nu\phi^\alpha g^{\mu\nu}) = 0 \tag{2}$$

whilst variation with respect to the metric $g_{\mu\nu}$ gives the equations

$$t_{\mu\nu} = 0 \tag{3}$$

where $t_{\mu\nu}$ is the three-dimensional energy–momentum tensor for the system. Equation (3) may be solved to identify $g_{\mu\nu}$ as the metric induced from the metric $\eta_{\alpha\beta}$ of space–time ($\eta_{\alpha\beta} = \text{diag}(-,+,+,+)$)

$$g_{\mu\nu} = \partial_\mu\phi^\alpha\partial_\nu\phi^\beta\eta_{\alpha\beta}. \tag{4}$$

Alternatively one can pass to the Hamiltonian formalism and show that the resultant secondary constraints are indeed the constraints discussed in Collins and Tucker (1976a).

Our next step is to supersymmetrise the action (1). In addition to the position variable $\phi^\alpha(x)$ we introduce a set of fields $\psi^\alpha(x)$ each of which transforms as an SL(2, R) spinor and is an odd element of a Grassmann algebra. SL(2, R) is isomorphic to SU(1, 1) which is the covering group of SO(2, 1). To describe the gauge field geometrically we introduce the *vierbeins* (more precisely *dreibeins*) $e_\mu{}^a(x)$ satisfying

$$e_\mu{}^a\eta_{ab}e_\nu{}^b = g_{\mu\nu} \tag{5}$$

where

$$\eta_{ab} = \text{diag}(-,+,+)$$

and

$$\sqrt{-g} = \det(e_\mu{}^a) \equiv e.$$

In addition we introduce the Rarita–Schwinger field as an SL(2, R) vector-spinor $\chi_\mu(x)$ whose components are also odd elements of a Grassmann algebra. The action for a locally supersymmetric membrane is given in terms of these fields by

$$S = \int d^3x\, e\mathscr{L} \tag{6}$$

where

$$\mathscr{L} = -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\mathrm{i}\bar{\psi}\gamma^\mu D_\mu\psi + \tfrac{1}{2}\mathrm{i}\bar{\chi}_\mu\gamma^\nu\gamma^\mu\partial_\nu\phi\psi - \tfrac{1}{16}\bar{\psi}\psi\bar{\chi}_\nu\gamma^\mu\gamma^\nu\chi_\mu - (\mathrm{i}/8e)\epsilon^{\lambda\mu\rho}\bar{\chi}_\lambda\gamma_\mu\chi_\rho + \tfrac{1}{4}\mathrm{i}\bar{\psi}\psi + \tfrac{1}{2}.$$

In this expression the γ matrices are given by

$$\gamma^\mu(x) = e_a{}^\mu(x)\gamma^a \qquad \gamma^0 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \gamma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \gamma^2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{7}$$

The adjoint spinor is given by

$$\bar{\psi} = \psi^{\mathrm{T}}\gamma^0 \tag{8}$$

and the SO(2, 1) covariant derivative is

$$D_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}\gamma_a\omega^a{}_\mu\psi \tag{9}$$

where $\omega^{a}{}_{\mu}$ is the connection field. Since we have no free 'supergravity' in (6) we must specify the connection $\omega^{a}{}_{\mu}$ of the space in which we are working. We take it to be

$$\omega^{c}{}_{\mu} = -\tfrac{1}{4}\epsilon^{abc} e^{d}{}_{\mu}(\Sigma_{dab} - \Sigma_{abd} - \Sigma_{bda}) \qquad (10)$$

where

$$\Sigma_{cab} = e^{\rho}{}_{a} e^{\lambda}{}_{b}(\partial_{\lambda} e_{c\rho} - \partial_{\rho} e_{c\lambda} + \tfrac{1}{2}\mathrm{i}\bar{\chi}_{\rho}\gamma_{c}\chi_{\lambda}). \qquad (11)$$

The presence of the bilinear term in χ in (11) implies that our three-dimensional manifold has torsion given by

$$T^{a}{}_{\mu\nu} = \mathrm{D}_{\mu} e^{a}{}_{\nu} - \mathrm{D}_{\nu} e^{a}{}_{\mu} = \tfrac{1}{2}\mathrm{i}\bar{\chi}_{\mu}\gamma^{a}\chi_{\nu}. \qquad (12)$$

We observe that, regarded as a field theory in a curved space, the membrane action (1) has a 'cosmological term'. To supersymmetrise the theory we have therefore elicited the aid of a 'supercosmological term'. In free four-dimensional supergravity this is given by a mass-type term for the χ_{μ} field (Freedman and Das 1977). In our case we find that we need not only a similar term but also a mass-type term for ψ. The action (6) is clearly SO(2, 1) and reparametrisation invariant. It is also invariant under the following local supergauge transformations:

$$\begin{aligned} &\delta\phi = \mathrm{i}\bar{\alpha}\psi && \delta\psi = \gamma^{\rho}(\partial_{\rho}\phi - \tfrac{1}{2}\mathrm{i}(\bar{\chi}_{\rho}\psi))\alpha \\ &\delta e^{a}{}_{\mu} = \mathrm{i}\bar{\alpha}\gamma^{a}\chi_{\mu} && \delta\chi_{\mu} = 2\mathrm{D}_{\mu}\alpha + \gamma_{\mu}\alpha. \end{aligned} \qquad (13)$$

We require, in addition, a constraint on the 'background' supergravity:

$$\epsilon^{\lambda\mu\rho}(\mathrm{D}_{\mu}\chi_{\rho} + \tfrac{1}{2}\gamma_{\mu}\chi_{\rho}) = 0. \qquad (14)$$

This condition need not be interpreted as a gauge restriction, as variation of (14) by a supergauge transformation implies

$$F^{a}{}_{\mu\nu} = e^{b}{}_{\mu} e^{c}{}_{\nu}\epsilon^{a}{}_{bc} - \tfrac{1}{2}\mathrm{i}\bar{\chi}_{\mu}\gamma^{a}\chi_{\nu} \qquad (15)$$

where

$$F^{a}{}_{\mu\nu} = -\tfrac{1}{2}\epsilon^{a}{}_{bc} R^{bc}{}_{\mu\nu} \qquad (16)$$

and $R^{bc}{}_{\mu\nu}$ is the curvature tensor. Hence, if the background supergravity satisfies (14) and (15), the action (6) possesses local supersymmetry†.

A more detailed study of this system together with a discussion of supersymmetry in three dimensions will be presented elsewhere (Howe and Tucker 1977a, b).

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† It may be noted that a similar condition must be imposed in order that the supergauge algebra for the spinning string should close in a field-dependent sense (Freedman and van Nieuwenhuizen 1976).

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